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*Published in:*  
18th Symposium on Mathematical Theory of Networks and Systems

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*Document Version*  
Publisher's PDF, also known as Version of record

*Publication date:*  
2008

[Link to publication in University of Groningen/UMCG research database](#)

*Citation for published version (APA):*

Jayawardhana, B., Logemann, H., & Ryan, E. P. (2008). The Circle Criterion and Input-to-State Stability for Infinite-Dimensional Systems. In *18th Symposium on Mathematical Theory of Networks and Systems* University of Groningen, Research Institute of Technology and Management.

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# The Circle Criterion and Input-to-State Stability for Infinite-Dimensional Systems<sup>\*</sup>

*B. Jayawardhana<sup>†</sup>, H. Logemann<sup>‡</sup>, and E.P. Ryan<sup>§</sup>*

## 1 Introduction

In this paper, the focus is on absolute stability and input-to-state stability of the feedback interconnection of an infinite-dimensional linear system  $\Sigma$  and a nonlinearity  $\Phi : \text{dom}(\Phi) \subset L^2_{\text{loc}}(\mathbb{R}_+, Y) \rightarrow L^2_{\text{loc}}(\mathbb{R}_+, U)$ , where  $\text{dom}(\Phi)$  denotes the domain of  $\Phi$  and  $U$  and  $Y$  (Hilbert spaces) denote the input and output spaces of  $\Sigma$ , respectively (see Figure 1, wherein  $v$  is an essentially bounded input signal). The system  $\Sigma$  is assumed to belong to the rather general class of well-posed systems (see, for example, [11, 13] and the references therein) and the nonlinearity is assumed to satisfy a (generalized) sector condition.

In the literature on the circle criterion for infinite-dimensional systems (see, for example, [3, 4, 5, 7, 9, 12], and the references therein), the emphasis is usually on  $L^2$ - or  $L^\infty$ -stability and global asymptotic or global exponential stability (or some variants thereof) of feedback systems of the type shown in Figure 1, with a static sector-bounded nonlinearity  $\Phi$  in the feedback path. The new contribution of this paper as compared to the previous literature is twofold.

(i) In addition to static nonlinearities, we include a class of dynamic nonlinearities which may exhibit bias, but still satisfy a generalized pointwise sector condition.

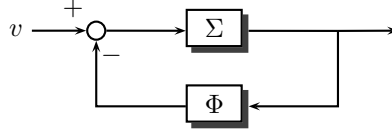
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<sup>\*</sup>Based on research supported by the UK Engineering & Physical Sciences Research Council (Grant Ref: GR/S94582/01).

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**Figure 1.** Feedback interconnection of linear system  $\Sigma$  and nonlinearity  $\Phi$

As specific subclasses, the class of nonlinearities encompasses both static nonlinearities with “negative resistance” and a wide range of hysteretic effects described by so-called Preisach operators.

(ii) The main results of the paper guarantee input-to-state-stability with “bias” (and “standard” input-to-state-stability if the nonlinearity is unbiased), thereby making contact with the important and rapidly developing input-to-state-stability theory in (finite-dimensional) nonlinear control.

As in the classical theory of absolute stability and circle criteria, the methodology involves a “symbiosis” of (generalized) sector data relating to the nonlinearity  $\Phi$  and properties of the transfer function of the linear system  $\Sigma$  to conclude stability properties of the feedback interconnection. We mention that the viewpoint of this paper is similar in spirit to that of [1]: however, the class of feedback systems considered here is very different to that in [1] as is the methodology adopted.

For sake of brevity, this paper does not contain any proofs: for these we refer to [6].

**Notation and terminology.** For  $\alpha \in \mathbb{R}$ , set  $\mathbb{C}_\alpha := \{s \in \mathbb{C} : \operatorname{Re} s > \alpha\}$ . If  $S$  is a non-empty subset of  $\mathbb{C}$ , then a set  $R \subset S$  is said to be *discrete* in  $S$ , if, for every  $s \in S$ , there exists a neighbourhood  $N$  of  $s$  such that  $N \cap R$  is finite. For Hilbert spaces  $U$  and  $Y$ , let  $\mathcal{B}(U, Y)$  denote the space of all linear bounded operators mapping  $U$  to  $Y$ . We write  $\mathcal{B}(U)$  for  $\mathcal{B}(U, U)$ . For  $T \in \mathcal{B}(U)$ , we define

$$\operatorname{Re} T := \frac{1}{2}(T + T^*) \in \mathcal{B}(U).$$

The space of all holomorphic and bounded functions  $\mathbb{C}_\alpha \rightarrow \mathcal{B}(U, Y)$  is denoted by  $H_\alpha^\infty(\mathcal{B}(U, Y))$ . We write  $H^\infty(\mathcal{B}(U, Y))$  for  $H_0^\infty(\mathcal{B}(U, Y))$ . Moreover, in the scalar case (that is  $U = Y = \mathbb{C}$ ), we simply write  $H_\alpha^\infty$ , or, if  $\alpha = 0$ ,  $H^\infty$  for  $H_\alpha^\infty(\mathcal{B}(U, Y))$  and  $H^\infty(\mathcal{B}(U, Y))$ , respectively. For  $\alpha \in \mathbb{R}$ , we define the exponentially weighted  $L^p$ -space  $L_\alpha^p(\mathbb{R}_+, X) := \{f \in L_{\text{loc}}^p(\mathbb{R}_+, U) : f(\cdot) \exp(-\alpha \cdot) \in L^p(\mathbb{R}_+, U)\}$ . The Laplace transform is denoted by  $\mathcal{L}$ .

## 2 Well-posed linear systems with nonlinear feedback

There are a number of equivalent definitions of well-posed systems, see, for example, [11, 13] and the references therein. We will be brief in the following and refer the reader to the literature for more details. Throughout, we shall be considering a well-posed system  $\Sigma$  with state-space  $X$ , input space  $U$  and output space  $Y$ , generating operators  $(A, B, C)$ , input-output operator  $G$  and transfer function  $\mathbf{G}$ . Here  $X$ ,  $U$  and  $Y$  are separable (complex) Hilbert spaces,  $A$  is the generator of a strongly

continuous semigroup  $\mathbf{T} = (\mathbf{T}_t)_{t \geq 0}$  on  $X$  and  $B \in \mathcal{B}(U, X_{-1})$  and  $C \in \mathcal{B}(X_1, Y)$ , respectively, are admissible control and observations for  $\mathbf{T}$ . The spaces  $X_1$  and  $X_{-1}$ , respectively, are interpolation and extrapolation spaces associated with  $X$ :  $X_1 = \text{dom}(A)$  (the domain of  $A$ ), endowed with the graph norm of  $A$ , whilst  $X_{-1}$  denotes the completion of  $X$  with respect to the norm  $\|x\|_{-1} = \|(\xi I - A)^{-1}x\|$ , where  $\xi \in \rho(A)$ , the resolvent set of  $A$  (different choices of  $\xi$  lead to equivalent norms) and  $\|\cdot\|$  denotes the norm on  $X$ . The control operator  $B$  is said to be *bounded* if it is so as a map from the input space  $U$  to the state space  $X$ , otherwise is said to be *unbounded*; the observation operator  $C$  is said to be *bounded* if it can be extended continuously to  $X$ , otherwise,  $C$  is said to be *unbounded*.

The so-called  $\Lambda$ -extension  $C_\Lambda$  of  $C$  is defined by

$$C_\Lambda z = \lim_{s \rightarrow \infty, s \in \mathbb{R}} C s(sI - A)^{-1} z,$$

with  $\text{dom}(C_\Lambda)$  (the domain of  $C_\Lambda$ ) consisting of all  $z \in X$  for which the above limit exists. The transfer function  $\mathbf{G}$  has the property that  $\mathbf{G} \in H^\infty(\mathcal{B}(U, Y))$  for every  $\omega > \omega(\mathbf{T})$ , where  $\omega(\mathbf{T})$  denotes the exponential growth constant of  $\mathbf{T}$ . Moreover, the input-output operator  $G : L^2_{\text{loc}}(\mathbb{R}_+, U) \rightarrow L^2_{\text{loc}}(\mathbb{R}_+, Y)$  is continuous and shift-invariant; for every  $\omega > \omega(\mathbf{T})$ ,  $G \in \mathcal{B}(L^2_\omega(\mathbb{R}_+, U), L^2_\omega(\mathbb{R}_+, Y))$  and

$$(\mathcal{L}(Gu))(s) = \mathbf{G}(s)(\mathcal{L}(u))(s), \quad \forall s \in \mathbb{C}_\omega, \quad \forall u \in L^2_\omega(\mathbb{R}_+, U).$$

In the following, let  $s_0 \in \mathbb{C}_{\omega(\mathbf{T})}$  be fixed, but arbitrary. For  $x^0 \in X$  and  $u \in L^2_{\text{loc}}(\mathbb{R}_+, U)$ , let  $x$  and  $y$  denote the state and output functions of  $\Sigma$ , respectively, corresponding to the initial condition  $x(0) = x^0 \in X$  and the input function  $u$ . Then  $x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} B u(\tau) d\tau$  for all  $t \in \mathbb{R}_+$ ,  $x(t) - (s_0 I - A)^{-1} B u(t) \in \text{dom}(C_\Lambda)$  for a.e.  $t \in \mathbb{R}_+$  and

$$\begin{aligned} \dot{x}(t) &= A x(t) + B u(t), \quad x(0) = x^0, \quad \text{a.e. } t \in \mathbb{R}_+, \\ y(t) &= C_\Lambda (x(t) - (s_0 I - A)^{-1} B u(t)) + \mathbf{G}(s_0) u(t), \quad \text{a.e. } t \geq 0. \end{aligned} \tag{1}$$

Of course, the differential equation in (1) has to be interpreted in  $X_{-1}$ . In the following, we identify  $\Sigma$  and (1) and refer to (1) as a well-posed system.

We say that (1) is *exponentially stable* if  $\omega(\mathbf{T}) < 0$  and we say that (1) is *input-output stable* if  $\mathbf{G} \in H^\infty(\mathcal{B}(U, Y))$  or, equivalently, if  $G \in \mathcal{B}(L^2(\mathbb{R}_+, U), L^2(\mathbb{R}_+, Y))$ . Furthermore, (1) is said to be *optimizable*, if for every  $x^0$ , there exists  $u \in L^2(\mathbb{R}_+, U)$  such that the function  $t \mapsto \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} B u(\tau) d\tau$  is in  $L^2(\mathbb{R}_+, X)$ . Writing  $X_{-1}^* := (X^*)_{-1}$ , we have that  $X_{-1}^* = (X_1)^*$  and  $C^* \in \mathcal{B}(Y, X_{-1}^*)$  is an admissible control operator for the adjoint semigroup  $\mathbf{T}^* = (\mathbf{T}_t^*)_{t \geq 0}$ . We say that (1) is *estimatable* if for every  $x^0$ , there exists  $u^* \in L^2(\mathbb{R}_+, Y)$  such the function  $t \mapsto \mathbf{T}_t^* x^0 + \int_0^t \mathbf{T}_{t-\tau}^* C^* u^*(\tau) d\tau$  is in  $L^2(\mathbb{R}_+, X)$ .

In the following, we will consider the closed-loop system obtained by applying the nonlinear feedback

$$u = v - \Phi(y) \tag{2}$$

to the well-posed linear system (1), where  $v \in L^\infty(\mathbb{R}_+, U)$  and the nonlinear operator  $\Phi : \text{dom}(\Phi) \subset L^2_{\text{loc}}(\mathbb{R}_+, Y) \rightarrow L^2_{\text{loc}}(\mathbb{R}_+, U)$  is causal. To define the concept

of a (local) solution of the feedback system given by (1) and (2), we first need to show that  $\Phi$  can be “localized” in the sense that it can be “extended” to spaces of functions with a finite time horizon. To this end, let  $0 < \sigma \leq \infty$  be arbitrary and set

$$\text{dom}_\sigma(\Phi) := \{w \in L^2_{\text{loc}}([0, \sigma), Y) : \forall \tau \in (0, \sigma) \exists w_\tau \in \text{dom}(\Phi) \text{ s.t. } w = w_\tau \text{ on } [0, \tau]\}.$$

Trivially,  $\text{dom}_\infty(\Phi) = \text{dom}(\Phi)$ . For  $w \in \text{dom}_\sigma(\Phi)$  with  $\sigma < \infty$ , we define  $\Phi(w)$  by

$$(\Phi(w))(t) = (\Phi(w_\tau))(t), \quad 0 \leq t \leq \tau < \sigma,$$

where  $w_\tau \in \text{dom}(\Phi)$  such that  $w = w_\tau$  on  $[0, \tau]$ . By causality of  $\Phi$ , this definition does not depend on the choice of  $\tau$  and thus  $\Phi(w)$  is a well-defined element in  $L^2_{\text{loc}}([0, \sigma), U)$ .

A *solution* on  $[0, \sigma)$  (where  $0 < \sigma \leq \infty$ ) of the feedback system given by (1) and (2) is a pair  $(x, y) \in C([0, \sigma), X) \times \text{dom}_\sigma(\Phi)$  such that, with  $u$  given by (2),

$$x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} B u(\tau) d\tau, \quad \forall t \in [0, \sigma), \quad (3)$$

$$y(t) = C_\Lambda (x(t) - (s_0 I - A)^{-1} B u(t)) + \mathbf{G}(s_0) u(t), \quad \text{a.e. } t \in [0, \sigma). \quad (4)$$

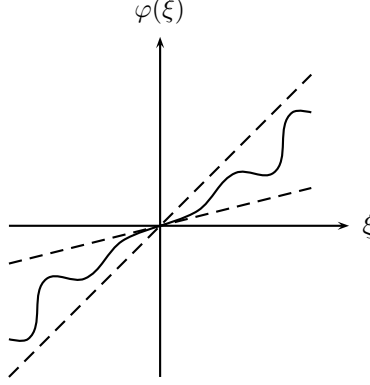
If  $\sigma = \infty$ , then we say that  $(x, y)$  is a *global solution*. Let  $\mathcal{S}$  denote the set of all  $(x^0, v) \in X \times L^\infty(\mathbb{R}_+, U)$  for which the feedback system given by (1) and (2) has at least one global solution. If  $(x^0, v) \in \mathcal{S}$ , then the notation  $(x(\cdot; x^0, v), y(\cdot; x^0, v))$  is used to denote any global solution corresponding to the initial condition  $x^0$  and the closed-loop input  $v$ . Furthermore, a routine argument based on Zorn’s lemma shows that every solution  $(x, y)$  can be extended to a *maximal solution*, that is, to a maximally defined solution which cannot be extended any further. The interval on which a maximal solution is defined is called the *maximal interval of existence* of the solution. We say that the feedback system given by (1) and (2) has the *blow-up property* if for every maximal solution  $(x, y)$  defined on a finite maximal interval of existence  $[0, \sigma)$ , the  $L^2$ -norm of  $y$  blows up, that is,  $\|y\|_{L^2(0, \tau)} \rightarrow \infty$  as  $\tau \uparrow \sigma$ . In this paper, we are mainly concerned with stability properties of the feedback system given by (1) and (2): whilst of fundamental importance, the question of existence of solutions is not the main concern here; this question requires addressing on a less general basis, taking into account relevant features of the particular system or subclass of systems under consideration (see [6] for further comments in this context).

### 3 The sector condition and input-to-state stability

First, we introduce a sector condition on the class of nonlinearities (in due course, this condition will be weakened to a generalized sector condition).

**Definition 1.** A nonlinearity  $\Phi : \text{dom}(\Phi) \subset L^2_{\text{loc}}(\mathbb{R}_+, Y) \rightarrow L^2_{\text{loc}}(\mathbb{R}_+, U)$  satisfies a *sector condition* if there exist operators  $K_1, K_2 \in \mathcal{B}(Y, U)$  such that

$$\text{Re} \langle (\Phi(w))(t) - K_1 w(t), (\Phi(w))(t) - K_2 w(t) \rangle \leq 0, \quad \forall w \in \text{dom}(\Phi), \text{ a.e. } t \in \mathbb{R}_+. \quad (5)$$



**Figure 2.** Sector-bounded static nonlinearity  $\varphi$

**Example 2 (Static nonlinearities).** Let  $\varphi : Y \rightarrow U$  be continuous and assume that there exist  $K_1, K_2 \in \mathcal{B}(Y, U)$  such that

$$\operatorname{Re} \langle \varphi(\xi) - K_1 \xi, \varphi(\xi) - K_2 \xi \rangle_U \leq 0 \quad \forall \xi \in Y. \quad (6)$$

With  $\varphi$  we may associate the Nemyckii operator  $\Phi : L_{\text{loc}}^2(\mathbb{R}_+, Y) \rightarrow L_{\text{loc}}^2(\mathbb{R}_+, U)$ , defined by  $\Phi(w) := \varphi \circ w$ . This operator satisfies the sector condition (5). Such operators provide a simple prototype class for the general nonlinearities considered in this section: at the simplest illustrative level, static sector-bounded scalar nonlinearities  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  of the type shown in Figure 2 (ubiquitous in the literature on the classical circle criterion) are subsumed by the formulation. This observation extends *mutatis mutandis* to encompass time-dependent static nonlinearities  $\varphi : \mathbb{R}_+ \times Y \rightarrow U$ .  $\diamond$

Anticipating Sections 4 and 5 below, we will also consider static nonlinearities for which the inequality in (6) is assumed to hold only outside some bounded set  $E \subset Y$  (see Figure 3). To accommodate these and more general nonlinearities, in Section 4 we will introduce a generalized sector condition and remark here that the generalized formulation encompasses a large class of hysteresis operators, including hysteresis of Preisach type.

Let  $K_1, K_2 \in \mathcal{B}(Y, U)$  and define

$$K := \frac{1}{2}(K_1 + K_2), \quad \kappa := \|K_2 - K_1\|^2. \quad (7)$$

We assemble the following hypotheses on the transfer function  $\mathbf{G}$  of (1) which will be variously invoked in the theory presented below.

**(H1)** There exists  $\alpha < 0$  and an open set  $\Omega \subset \mathbb{C}_\alpha$  such that  $\mathbb{C}_\alpha \setminus \Omega$  is discrete in

$\mathbb{C}_\alpha$  and  $\mathbf{G}$  is holomorphic on  $\Omega$ , the frequency-domain condition

$$\mathbf{G}^*(i\omega) \left[ \frac{\kappa + \delta}{4} I - K^* K \right] \mathbf{G}(i\omega) \leq I + 2 \operatorname{Re}(K \mathbf{G}(i\omega)), \quad \text{a.e. } \omega \in \mathbb{R}. \quad (8)$$

holds for some  $\delta > 0$  and  $\mathbf{G}(I + K \mathbf{G})^{-1} \in H^\infty(\mathcal{B}(U, Y))$ ,

(H2)  $\mathbf{G} \in H^\infty(\mathcal{B}(U, Y))$  and there exist  $\delta > 0$  and  $\rho < 1$  such that (8) holds and

$$\mathbf{G}^*(i\omega) \left[ \frac{\kappa + \delta}{4} I - K^* K \right] \mathbf{G}(i\omega) \geq -\rho I, \quad \text{a.e. } \omega \in \mathbb{R}. \quad (9)$$

(H3) There exists an open set  $\Omega \subset \mathbb{C}_0$  such that  $\mathbb{C}_0 \setminus \Omega$  is discrete in  $\mathbb{C}_0$  and  $\mathbf{G}$  is holomorphic on  $\Omega$ ,  $I + K \mathbf{G}(s)$  is invertible for all  $s \in \Omega$  and the frequency-domain condition

$$\mathbf{G}^*(s) \left[ \frac{\kappa + \delta}{4} I - K^* K \right] \mathbf{G}(s) \leq I + 2 \operatorname{Re}(K \mathbf{G}(s)), \quad \forall s \in \Omega \quad (10)$$

holds for some  $\delta > 0$ .

(H4) There exists an open set  $\Omega \subset \mathbb{C}_0$  such that  $\mathbb{C}_0 \setminus \Omega$  is discrete in  $\mathbb{C}_0$  and  $\mathbf{G}$  is holomorphic on  $\Omega$ ,  $K \mathbf{G}(s)$  is compact for all  $s \in \Omega$  and the frequency-domain condition (10) holds for some  $\delta > 0$ .

**Remark 3.** (a) In the case of scalar “sector data”, that is  $U = Y$  and there exist  $k_1, k_2 \in \mathbb{C}$  such that  $K_1 = k_1 I$  and  $K_2 = k_2 I$ , the term

$$\frac{\kappa + \delta}{4} I - K^* K$$

appearing on the left-hand sides of (8)-(10) simplifies to  $(\delta/4 - \operatorname{Re}(\bar{k}_1 k_2))I$ .

(b) Assume that one of the operators  $K_1$  and  $K_2$  is the zero operator and that the other is a scalar multiple of an isometry. Then it is not difficult to show that (H2) is satisfied, provided that  $\mathbf{G} \in H^\infty(\mathcal{B}(U, Y))$  and the positive-real condition

$$\varepsilon I \leq I + 2 \operatorname{Re}(K \mathbf{G}(i\omega)), \quad \text{a.e. } \omega \in \mathbb{R}$$

holds for some  $\varepsilon > 0$ .

We are now in the position to state the main result of this section.

**Theorem 4.** *Assume that (1) is optimizable and estimatable and that there exist operators  $K_1, K_2 \in \mathcal{B}(Y, U)$  such that  $\Phi$  satisfies the sector condition (5). Let  $K \in \mathcal{B}(Y, U)$  and  $\kappa \geq 0$  be given by (7). If at least one of hypotheses (H1)–(H4) holds, then there exist positive constants  $\Gamma$  and  $\gamma$ , such that, for each  $(x^0, v) \in \mathcal{S}$ ,*

$$\|x(t; x^0, v)\| \leq \Gamma (\exp(-\gamma t) \|x^0\| + \|v\|_{L^\infty}), \quad \forall t \in \mathbb{R}_+. \quad (11)$$

For the above theorem to be non-vacuous,  $\mathcal{S}$  should be non-empty: thus, there is a tacit assumption of global existence of solutions. However, if the feedback

system given by (1) and (2) has the blow-up property, then it can be shown that the assumptions of Theorem 4 imply that every (local) solution can be extended to a global solution. Furthermore, we emphasize that (11) implies in particular that the feedback system is input-to-state stable in the sense of Sontag (see [10] for a recent survey of the theory of input-to-state stability).

Theorem 4 can be considered as a generalization and refinement of the circle criterion (see, for example, [4, 12]): in particular, it shows that, under the standard assumptions of the circle criterion (see also Corollaries 5 and 6 below), input-to-state stability is guaranteed. The proof of Theorem 4 (see [6]) is based on a well-known exponential weighting technique which has been used to prove stability results of input-output type (see [4, Section V.3] and the references therein). The application of this technique in an input-to-state stability context seems to be new (even in the finite-dimensional case). In particular, whilst the standard text-book version of the circle criterion for finite-dimensional state-space systems is usually proved using Lyapunov techniques combined with the positive-real lemma (see, for example, [12, p. 227]), the approach based on the exponential weighting technique provides a more elementary alternative.

The following corollary considers the case of scalar “sector data”.

**Corollary 5.** *Assume that (1) is optimizable and estimatable,  $U = Y$  and that there exists an open set  $\Omega \subset \mathbb{C}_0$  such that  $\mathbb{C}_0 \setminus \Omega$  is discrete in  $\mathbb{C}_0$  and  $\mathbf{G}$  is holomorphic on  $\Omega$ . Furthermore, assume that there exist  $k_1, k_2 \in \mathbb{C}$  and  $\varepsilon > 0$  such that  $\Phi$  satisfies (5) with  $K_1 = k_1 I$  and  $K_2 = k_2 I$ ,  $I + k_1 \mathbf{G}(s)$  is invertible for every  $s \in \Omega$  and*

$$\operatorname{Re}[(I + k_2 \mathbf{G}(s))(I + k_1 \mathbf{G}(s))^{-1}] \geq \varepsilon I, \quad \forall s \in \Omega. \quad (12)$$

*Then there exist positive constants  $\Gamma$  and  $\gamma$ , such that, for each  $(x^0, v) \in \mathcal{S}$ , (11) holds.*

For non-zero real numbers  $k_1$  and  $k_2$ , we define

$$\Delta(k_1, k_2) := \text{open disk in } \mathbb{C} \text{ with centre in } \mathbb{R} \text{ and } -\frac{1}{k_1} \text{ and } -\frac{1}{k_2} \text{ in its boundary.}$$

The next corollary focuses on the single-input-single-output case. In particular, the classical circle criterion is recovered.

**Corollary 6.** *Assume that (1) is optimizable and estimatable,  $U = Y = \mathbb{R}$  and there exist real numbers  $k_1 < k_2$  such that*

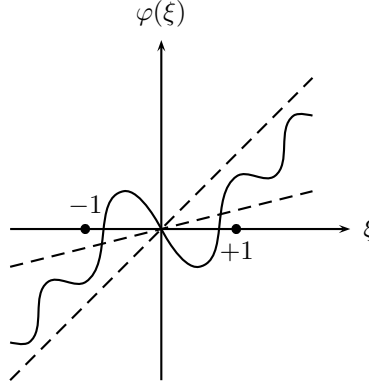
$$((\Phi(w))(t) - k_1 w(t))((\Phi(w))(t) - k_2 w(t)) \leq 0, \quad \forall w \in \operatorname{dom}(\Phi), \text{ a.e. } t \in \mathbb{R}_+. \quad (13)$$

*Then there exist positive constants  $\Gamma$  and  $\gamma$ , such that, for each  $(x^0, v) \in \mathcal{S}$ , (11) holds, provided that one of the following conditions is satisfied:*

**(C1)**  $0 < k_1 < k_2$ ,  $\mathbf{G}/(1 + [(k_1 + k_2)/2]\mathbf{G}) \in H^\infty$ ,  $\mathbf{G}(i\omega)$  is bounded away from  $\Delta(k_1, k_2)$  for all  $\omega \in \mathbb{R}$  for which  $i\omega$  is not a pole of  $\mathbf{G}$ ;

**(C2)**  $0 = k_1 < k_2$ ,  $\mathbf{G} \in H^\infty$  and there exists  $\delta > 0$  such that  $1 + k_2 \operatorname{Re} \mathbf{G}(i\omega) \geq \delta$  for all  $\omega \in \mathbb{R}$ ;





**Figure 3.** Static nonlinearity  $\varphi$  satisfying a generalized sector condition

**(C3)**  $k_1 < 0 < k_2$ ,  $\mathbf{G} \in H^\infty$ ,  $\mathbf{G}(i\omega) \in \Delta(k_1, k_2)$  for all  $\omega \in \mathbb{R}$  and  $\mathbf{G}(i\omega)$  is bounded away from  $\partial\Delta(k_1, k_2)$  for all  $\omega \in \mathbb{R}$ .

Observe that, in this single-input-single-output setting, the sector condition (13) can be expressed in the equivalent form:

$$k_1 w^2(t) \leq (\Phi(w))(t)w(t) \leq k_2 w^2(t), \quad \forall w \in \text{dom}(\Phi), \text{ a.e. } t \in \mathbb{R}_+. \quad (14)$$

In many situations, the input-output stability condition  $\mathbf{G}/(1 + [(k_1 + k_2)/2]\mathbf{G}) \in H^\infty$  (imposed in (C1)) is satisfied, provided that the number of anticlockwise encirclements of  $\Delta(k_1, k_2)$  by the Nyquist diagram of  $\mathbf{G}$  is equal to the number of poles of  $\mathbf{G}$  in  $\mathbb{C}_0$ , see, for example, [4, 12].

## 4 Generalized sector condition and input-to-state stability with bias

Next, we seek to relax the condition (5) to a *generalized sector condition*. Loosely speaking, we wish to impose the (pointwise) inequality in (5) only when  $t \in \mathbb{R}_+$  and  $w \in \text{dom}(\Phi)$  are such that  $w(t) \in Y \setminus E$ , where  $E$  (the exceptional set) is some bounded subset of  $Y$ . A prototype to bear in mind is the case wherein  $\Phi$  is the Nemyckii operator, given by  $\Phi(w) := \varphi \circ w$ , associated with a static nonlinearity  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ , of the form shown in Figure 3 (a nonlinearity with negative resistance), satisfying a sector condition outside the interval  $E = [-1, 1]$ . Extrapolating this prototype to our abstract setting requires care. The issue is to circumvent the technical difficulty engendered by the fact that the general operator  $\Phi$  has domain  $\text{dom}(\Phi) \subset L^2_{\text{loc}}(\mathbb{R}_+, Y)$  and so  $\Phi$  acts on equivalence classes of functions  $\mathbb{R}_+ \rightarrow Y$ . Let  $w \in L^2_{\text{loc}}(\mathbb{R}_+, Y)$  and  $Z \subset Y$  be arbitrary. Let  $w_r : \mathbb{R}_+ \rightarrow Y$  be any representative of  $w$  and denote the preimage of  $Z$  under  $w_r$  by  $w_r^{-1}(Z) := \{t \in \mathbb{R}_+ : w_r(t) \in Z\}$ . Let  $\mathbb{I}_{w_r^{-1}(Z)}$  be the indicator or characteristic function of the set  $w_r^{-1}(Z)$  and define  $\chi_Z(w) \in L^2_{\text{loc}}(\mathbb{R}_+, Y)$  to be the equivalence class of this function,

that is,

$$\chi_Z(w) := [\mathbb{I}_{w_r^{-1}(Z)}].$$

Every choice of representative  $w_r$  of  $w$  yields the same equivalence class  $[\mathbb{I}_{w_r^{-1}(Z)}]$  and so  $\chi_Z(w)$  is a well-defined element of  $L_{\text{loc}}^2(\mathbb{R}_+, Y)$  for all  $w \in L_{\text{loc}}^2(\mathbb{R}_+, Y)$ . We are now in a position to define the requisite generalized sector condition.

**Definition 7.** A nonlinearity  $\Phi : \text{dom}(\Phi) \subset L_{\text{loc}}^2(\mathbb{R}_+, Y) \rightarrow L_{\text{loc}}^2(\mathbb{R}_+, U)$  satisfies a *generalized sector condition* if there exist operators  $K_1, K_2 \in \mathcal{B}(Y, U)$ , a bounded set  $E \subset Y$  and a constant  $b \geq 0$  such that, for all  $w \in \text{dom}(\Phi)$  and a.e.  $t \in \mathbb{R}_+$ ,

$$\text{Re} \langle (\Phi(w))(t) - K_1 w(t), (\Phi(w))(t) - K_2 w(t) \rangle (\chi_{Y \setminus E}(w))(t) \leq 0 \quad (15)$$

and

$$\|(\Phi(w))(t)\| (\chi_E(w))(t) \leq b. \quad (16)$$

The following result generalizes Theorem 4.

**Corollary 8.** Assume that (1) is optimizable and estimatable and that there exist operators  $K_1, K_2 \in \mathcal{B}(Y, U)$ ,  $b \geq 0$  and a bounded set  $E \subset Y$  such that  $\Phi$  satisfies (15) and (16) for all  $w \in \text{dom}(\Phi)$  and a.e.  $t \in \mathbb{R}_+$ . Let  $K \in \mathcal{B}(Y, U)$  and  $\kappa \geq 0$  be given by (7). If at least one of hypotheses (H1)–(H4) holds, then there exist positive constants  $\Gamma$  and  $\gamma$  such that, for each  $(x^0, v) \in \mathcal{S}$ ,

$$\|x(t; x^0, v)\| \leq \Gamma (\exp(-\gamma t) \|x^0\| + \|v\|_{L^\infty} + \beta), \quad \forall t \in \mathbb{R}_+, \quad (17)$$

where

$$\beta := \sup \{ \|(\Phi(w) - Kw)\chi_E(w)\|_{L^\infty} : w \in \text{dom}(\Phi) \} \leq b + \sup_{\xi \in E} \|K\xi\|, \quad (18)$$

In particular, (17) provides an input-to-state stability estimate with bias  $\beta$  (input-to-state stability with bias  $\beta$ ). Under the additional assumption that the feedback system given by (1) and (2) has the blow-up property, it can be shown that the hypotheses of Corollary 8 imply that every maximal solution is global, so that every (local) solution can be extended to a global solution (to which then the stability conclusions of Corollary 8 apply).

The following results are generalizations of Corollaries 5 and 6.

**Corollary 9.** Assume that (1) is optimizable and estimatable,  $U = Y$  and that there exists an open set  $\Omega \subset \mathbb{C}_0$  such that  $\mathbb{C}_0 \setminus \Omega$  is discrete in  $\mathbb{C}_0$  and  $\mathbf{G}$  is holomorphic on  $\Omega$ . Furthermore, assume that there exist  $k_1, k_2 \in \mathbb{C}$ , a bounded set  $E \subset Y$  and constants  $b \geq 0$  and  $\varepsilon > 0$  such that, for all  $w \in \text{dom}(\Phi)$  and a.e.  $t \in \mathbb{R}_+$ ,  $\Phi$  satisfies (15) and (16) (with  $K_1 = k_1 I$  and  $K_2 = k_2 I$ ),  $I + k_1 \mathbf{G}(s)$  is invertible for every  $s \in \Omega$  and the positive-real condition

$$\text{Re}[(I + k_2 \mathbf{G}(s))(I + k_1 \mathbf{G}(s))^{-1}] \geq \varepsilon I, \quad \forall s \in \Omega$$

holds. Then there exist constants  $\Gamma > 0$  and  $\gamma > 0$  such that, for each  $(x^0, v) \in \mathcal{S}$ , (17) holds, where  $\beta \geq 0$  is given by (18).

**Corollary 10.** Assume that (1) is optimizable and estimatable,  $U = Y = \mathbb{R}$  and there exist real numbers  $k_1 < k_2$ , a bounded set  $E \subset \mathbb{R}$  and  $b \geq 0$  such that

$$((\Phi(w))(t) - k_1 w(t))((\Phi(w))(t) - k_2 w(t))(\chi_{Y \setminus E}(w))(t) \leq 0, \quad \forall w \in \text{dom}(\Phi), \text{ a.e. } t \in \mathbb{R}_+$$

and

$$|(\Phi(w))(t)|(\chi_E(w))(t) \leq b, \quad \forall w \in \text{dom}(\Phi), \text{ a.e. } t \in \mathbb{R}_+.$$

If at least one of the conditions (C1)–(C3) of Corollary 6 is satisfied, then there exist  $\Gamma > 0$  and  $\gamma > 0$  such that, for each  $(x^0, v) \in \mathcal{S}$ , (17) holds, where

$$\beta := \sup \left\{ \|(\Phi(w) - (k_1 + k_2)w/2)\chi_E(w)\|_{L^\infty} : w \in \text{dom}(\Phi) \right\} \leq b + |k_1 + k_2| \sup_{\xi \in E} |\xi|/2, \quad (19)$$

## 5 Hysteretic feedback systems

Consider again the feedback interconnection of Figure 1, but now in a single-input ( $U = \mathbb{R}$ ), single-output ( $Y = \mathbb{R}$ ) setting and with a hysteresis operator  $\Phi$  in the feedback path. An operator  $\Phi : C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$  is a *hysteresis operator* if it is causal and rate independent. Here *rate independence* means that  $\Phi(w \circ \zeta) = (\Phi w) \circ \zeta$  for every  $w \in C(\mathbb{R}_+)$  and every time transformation  $\zeta$ , where  $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to be a *time transformation* if it is continuous, non-decreasing and surjective.

For simplicity of presentation, henceforth we restrict attention to the class of Preisach hysteresis operators which model complex hysteresis effects: for example, nested loops in input-output characteristics. A basic building block for the Preisach operator is the hysteresis operator  $\mathcal{B}_{\sigma, \xi}$ , the so-called *backlash* operator with width  $\sigma \geq 0$  and “initial condition”  $\xi \in \mathbb{R}$ . A discussion of the backlash operator (also called *play* operator) can be found in a number of references, see for example, [2] and [8].

Let  $\xi : \mathbb{R}_+ \rightarrow \mathbb{R}$  be a compactly supported and globally Lipschitz function with Lipschitz constant 1. Let  $\mu$  be a regular signed Borel measure on  $\mathbb{R}_+$ . Denoting Lebesgue measure on  $\mathbb{R}$  by  $\mu_L$ , let  $f : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$  be a locally  $(\mu_L \otimes \mu)$ -integrable function and let  $f_0 \in \mathbb{R}$ . The operator  $\mathcal{P}_\xi : C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$  defined by

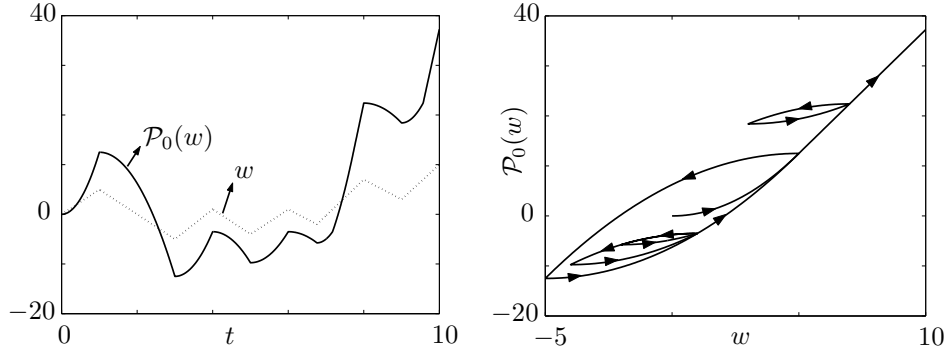
$$(\mathcal{P}_\xi(w))(t) = \int_0^\infty \int_0^{(\mathcal{B}_{\sigma, \xi(\sigma)}(w))(t)} f(s, \sigma) \mu_L(ds) \mu(d\sigma) + f_0 \quad \forall w \in C(\mathbb{R}_+), \quad \forall t \in \mathbb{R}_+, \quad (20)$$

is called a *Preisach* operator, cf. [2, p. 55]. It is well-known that  $\mathcal{P}_\xi$  is a hysteresis operator (this follows from the fact that  $\mathcal{B}_{\sigma, \xi(\sigma)}$  is a hysteresis operator for every  $\sigma \geq 0$ ).

Setting  $f(\cdot, \cdot) = 1$  and  $f_0 = 0$  in (20), we obtain the *Prandtl* operator  $\mathcal{P}_\xi : C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$  defined by

$$\mathcal{P}_\xi(w)(t) = \int_0^\infty (\mathcal{B}_{\sigma, \xi(\sigma)}(w))(t) \mu(d\sigma) \quad \forall w \in C(\mathbb{R}_+), \quad \forall t \in \mathbb{R}_+. \quad (21)$$

For  $\xi(\cdot) = 0$  and  $\mu$  given by  $\mu(S) = \int_S \mathbb{I}_{[0,5]}(\sigma) d\sigma$  (where  $\mathbb{I}_{[0,5]}$  denotes the indicator function of the interval  $[0, 5]$ ), the Prandtl operator is illustrated in Figure 4.



**Figure 4.** Example of Prandtl hysteresis

The next proposition identifies (rather “mild”) conditions under which the Preisach operator (20) satisfies a generalized sector bound and hence fits into the theory developed in Section 4. For simplicity, we assume that the measure  $\mu$  and the function  $f$  are non-negative (an important case in applications), although the proposition can be extended to signed measures  $\mu$  and sign-indefinite functions  $f$ .

**Proposition 11.** *Let  $\mathcal{P}_\xi$  be the Preisach operator defined in (20). Assume that the measure  $\mu$  is non-negative,  $a_1 := \mu(\mathbb{R}_+) < \infty$  and  $a_2 := \int_0^\infty \sigma \mu(d\sigma) < \infty$ . Furthermore, assume that*

$$b_1 := \operatorname{ess\,inf}_{(s,\sigma) \in \mathbb{R} \times \mathbb{R}_+} f(s, \sigma) \geq 0, \quad b_2 := \operatorname{ess\,sup}_{(s,\sigma) \in \mathbb{R} \times \mathbb{R}_+} f(s, \sigma) < \infty$$

and set

$$a_{\mathcal{P}} := a_1 b_1, \quad b_{\mathcal{P}} := a_1 b_2, \quad c_{\mathcal{P}} := a_2 b_2 + |f_0|. \quad (22)$$

Then, for all  $w \in C(\mathbb{R}_+)$  and all  $t \in \mathbb{R}_+$ ,

$$w(t) \geq 0 \implies a_{\mathcal{P}} w(t) - c_{\mathcal{P}} \leq (\mathcal{P}_\xi(w))(t) \leq b_{\mathcal{P}} w(t) + c_{\mathcal{P}}, \quad (23)$$

$$w(t) \leq 0 \implies b_{\mathcal{P}} w(t) - c_{\mathcal{P}} \leq (\mathcal{P}_\xi(w))(t) \leq a_{\mathcal{P}} w(t) + c_{\mathcal{P}}, \quad (24)$$

and, furthermore, for every  $\eta > 0$ ,

$$|w(t)| \geq c_{\mathcal{P}}/\eta \implies (a_{\mathcal{P}} - \eta)w^2(t) \leq (\mathcal{P}_\xi(w))(t)y(t) \leq (b_{\mathcal{P}} + \eta)w^2(t). \quad (25)$$

In particular, for every  $\eta > 0$ , the generalized sector conditions (15) and (16) hold with  $U = \mathbb{R} = Y$ ,  $E = [-c_{\mathcal{P}}/\eta, c_{\mathcal{P}}/\eta]$ ,  $K_1 = (a_{\mathcal{P}} - \eta)I$ ,  $K_2 = (b_{\mathcal{P}} + \eta)I$ , and  $b = (b_{\mathcal{P}}/\eta + 1)c_{\mathcal{P}}$ .

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